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On a Functional Differential Equation that Arises in a Markov Control Problem*

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1. INTRODUCTION

The purpose of this paper is to show the existence of a unique solution of a functional differential equation of advanced type that arises in connection with the problem of controlling a certain kind of Markov process known as a storage process. Actually, two cases are considered, as the interval S of the real line equals either $[0, s]$ (with $0 < s < \infty$) or $[0, \infty)$.

Let A denote a compact subset of n-dimensional Euclidean space. Let p and r denote two continuous, real-valued functions on $S \times A$ with, moreover, r nonnegative. Let λ be a positive constant. For each pair $(x, a) \in S \times A$, consider a nonnegative measure $\beta(x, a, \cdot)$ on the Borel subsets of S. The following assumptions will remain in effect about β: (i) for each pair $(x, a) \in S \times A$, $\beta(x, a, [0, x)) = 0$, (ii) $\beta(0, a, \{0\}) = 0$ for all $a \in A$, (iii) $\sup_{x,a} \beta(x, a, S) < \infty$, and (iv) if $x_n \to x$, $a_n \to a$, then $\beta(x_n, a_n, \cdot)$ converges weakly to $\beta(x, a, \cdot)$.

The following theorem is the main result for the first case.

THEOREM 1. *For the case $S = [0, s]$, suppose for each $x > 0$ that r is strictly positive on $[x, s] \times A$. Then there exists a unique, continuous real-valued function v on S that satisfies (with $v' \equiv dv/dx$)*

$$v'(x) = \sup_{a\in A} \left\{ r^{-1}(x, a) \left[\int [v(y) - v(x)]\, \beta(x, a, dy) - \lambda v(x) + p(x, a) \right] \right\}, \quad x > 0, \tag{1}$$

and the boundary condition

$$\sup_{a\in A} \left\{ \int [v(y) - v(0)]\, \beta(0, a, dy) - \lambda v(0) + p(0, a) \right\} = 0. \tag{2}$$

Moreover, v is continuously differentiable on $(0, s]$.

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The main result for the second case is as follows.

THEOREM 2. *For the case $S = [0, \infty)$, suppose p and r are bounded on $S \times A$ and that, for each $x > 0$, there exists some $\epsilon > 0$ such that $r \geqslant \epsilon$ on $[x, \infty) \times A$. In addition, suppose there exists some continuous function q on A such that $\lim_{x \to \infty} p(x, a) = q(a)$ for every $a \in A$. Then there exists a unique, bounded, continuous, real-valued function v on S satisfying* (1) *and* (2). *Moreover, this solution has a continuous derivative on* $(0, \infty)$.

The results for the first case are used for the second. To prove Theorem 1, the first step is to show there exists a solution on $[0, s]$ to (1) satisfying an initial condition at the boundary s. One approach to this step would be to utilize some of the ideas on functional differential equations in Hale [7], namely, first use continuity to show local existence and then use a Lipschitz condition to show uniqueness. It is preferable, however, to directly obtain a global existence and uniqueness result. This is accomplished in this paper by generalizing an ordinary differential equation approach based on a Lipschitz condition that is presented in Edwards [6] but was originally due to Bielecki [2]. Since this global approach is apparently new for functional differential equations, Section 2 will present this result in general terms so that it can be read independently of the remainder of this paper.

After applying the theoretical results of Section 2 to the first part of the proof of Theorem 1, one must depart from the standard theory on functional differential equations and deal with boundary condition (2). This is accomplished in Section 3. Then Theorem 1 together with a limiting argument are used in Section 4 to prove Theorem 2.

The Markov control problem which motivated this paper has received very little attention. Doshi [5] and Mitchell [9] looked at two special cases, but apparently nobody has tackled the general model corresponding to Eqs. (1) and (2) here. Actually, these equations are transformations of the standard dynamic programming functional equation which one obtains for the relevant Markov control problem. Section 5, the final section, explains this transformation and briefly relates Eqs. (1) and (2) with the Markov control problem. It should be mentioned that Morais [10] utilizes the results of this paper to present a general theory for controlled Markov storage processes.

2. A Global Existence Theorem for Functional Differential Equations

Hale [7] first uses continuity to show the local existence of a solution to a functional differential equation and then uses a Lipschitz condition to show uniqueness. In cases where the Lipschitz condition exists, however, it is more

Hence

$$e^{-\lambda L|t-t_0|}\, |[y_n - y](t)| \leqslant |\, z_n - z \,|\, e^{-\lambda L|t-t_0|} + \lambda^{-1}N(y_n - y)$$
$$\leqslant |\, z_n - z \,| + \lambda^{-1}N(y_n - y)$$

for all $t \in T$. In particular,

$$N(y_n - y) \leqslant |\, z_n - z \,| + \lambda^{-1}N(y_n - y),$$

so upon moving the term $\lambda^{-1}N(y_n - y)$ to the left-hand side of this inequality, one quickly obtains the desired result. ∎

3. Proof of Theorem 1

The proof of Theorem 1 proceeds in three steps: first, Theorem 3 is used to show there exists a unique solution of (1) and $v(s) = z$ on $[\epsilon, s]$ for any $\epsilon > 0$ and z; second, it is shown that z may be chosen so that such a solution exists on all of S and satisfies the boundary condition; finally, uniqueness is established. This section is comprised of several lemmas.

Lemma 5. *For any $\epsilon > 0$ and real number z, there exists a unique solution of* (1) *on $[\epsilon, s]$ satisfying $v(s) = z$.*

Proof. It suffices to verify the hypotheses of Theorem 3. Set $f(x, v)$ equal to the right-hand side of (1) and $t_0 = s$. Now the argument in f is a continuous function on $[\epsilon, s] \times F \times A$ by the assumptions in the Introduction, so f is continuous by a standard argument such as in Berge [1, pp. 115–116]. Moreover, by the same assumptions, and noting that there exists some $\delta > 0$ such that $r \geqslant \delta$ on $[\epsilon, s]$, f is bounded. Finally, to show the Lipschitz condition, take arbitrary $x \in [\epsilon, s]$, $v_1, v_2 \in F$, and assume, without loss of generality, that $f(x, v_1) \geqslant f(x, v_2)$. Suppose a_1 maximizes the argument in $f(x, v_1)$. Then

$$\begin{aligned}
&|f(x, v_1) - f(x, v_2)| \\
&\quad \leqslant f(x, v_1) - r^{-1}(x, a_1) \left\{ \int [v_2(y) - v_2(x)]\, \beta(x, a_1, dy) - \lambda v_2(x) + p(x, a_1) \right\} \\
&\quad = r^{-1}(x, a_1) \left\{ \int [v_1(y) - v_2(y)]\, \beta(x, a_1, dy) \right. \\
&\qquad \left. - [v_1(x) - v_2(x)] \int \beta(x, a_1, dy) - \lambda [v_1(x) - v_2(x)] \right\} \\
&\quad \leqslant r^{-1}(x, a_1) \left\{ \sup_{x \leqslant y \leqslant s} |\, v_1(y) - v_2(y) | \int \beta(x, a_1, dy) \right. \\
&\qquad \left. + \left[\lambda + \int \beta(x, a_1, dy) \right] |\, v_1(x) - v_2(x) | \right\} \\
&\quad \leqslant L \sup |\, v_1(y) - v_1(y)|
\end{aligned}$$

For arbitrary $y_1, y_2 \in F$,

$$\begin{aligned}
|[u(y_1) - u(y_2)](t)| &= \left| \int_{t_0}^{t} [f(\tau, y_1) - f(\tau, y_2)]\, d\tau \right| \\
&\leqslant \int_{t_0}^{t} | f(\tau, y_1) - f(\tau, y_2)|\, d\tau \\
&\leqslant L \int_{t_0}^{t} \sup_{(t_0 \wedge \tau) \leqslant s < (t_0 \vee \tau)} | y_1(s) - y_2(s)|\, d\tau \\
&\leqslant LN(y_1 - y_2) \int_{t_0}^{t} e^{\lambda L |\tau - t_0|}\, d\tau \\
&\leqslant LN(y_1 - y_2)(\lambda L)^{-1} e^{\lambda L |t - t_0|} \\
&= \lambda^{-1} N(y_1 - y_2)\, e^{\lambda L |t - t_0|}
\end{aligned}$$

To see the third inequality, let $\bar{s}$ maximize $| y_1(s) - y_2(s)|$ over $[(t_0 \wedge \tau), (t_0 \vee \tau)]$. Then by the definition of N,

$$\begin{aligned}
N(y_1 - y_2) &\geqslant e^{-\lambda L |\bar{s} - t_0|} \, | y_1(\bar{s}) - y_2(\bar{s})| \\
&\geqslant e^{-\lambda L |\tau - t_0|} \, | y_1(\bar{s}) - y_2(\bar{s})|.
\end{aligned}$$

The last inequality follows because the integral is majorized by $(\lambda L)^{-1}[e^{\lambda L |t - t_0|} - 1]$. Hence

$$e^{-\lambda L |t - t_0|} \, |[u(y_1) - u(y_2)](t)| \leqslant \lambda^{-1} N(y_1 - y_2)$$

for all $t \in T$. It follows that u is a contraction, since $\lambda > 1$, so this proof is completed. ▮

According to Hale [7, p. 21], the solution of (3) depends in a continuous manner upon the initial condition z. This result can be shown in a simple manner with the approach used for Theorem 3. The following will be used later in this paper.

COROLLARY 4. *Consider a sequence $z_n \to z$. Let $y_n(y)$ denote the solution of* (3) *corresponding to z_n (respectively, z). Then $y_n \to y$ in F as $n \to \infty$.*

Proof. It suffices to show convergence in E, the Banach space defined in the proof of Theorem 3. In fact, according to that proof,

$$\begin{aligned}
|[y_n - y](t)| &\leqslant | z_n - z | + L \int_{t_0}^{t} \sup_{(t_0 \wedge \tau) \leqslant s \leqslant (t_0 \vee \tau)} | y_n(s) - y(s)|\, d\tau \\
&\leqslant | z_n - z | + \lambda^{-1} N(y_n - y)\, e^{\lambda L |t - t_0|}.
\end{aligned}$$

Hence

$$e^{-\lambda L|t-t_0|}\,|[y_n - y](t)| \leqslant |z_n - z|\, e^{-\lambda L|t-t_0|} + \lambda^{-1}N(y_n - y)$$
$$\leqslant |z_n - z| + \lambda^{-1}N(y_n - y)$$

for all $t \in T$. In particular,

$$N(y_n - y) \leqslant |z_n - z| + \lambda^{-1}N(y_n - y),$$

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Lemma 5. *For any $\epsilon > 0$ and real number z, there exists a unique solution of (1) on $[\epsilon, s]$ satisfying $v(s) = z$.*

Proof. It suffices to verify the hypotheses of Theorem 3. Set $f(x, v)$ equal to the right-hand side of (1) and $t_0 = s$. Now the argument in f is a continuous function on $[\epsilon, s] \times F \times A$ by the assumptions in the Introduction, so f is continuous by a standard argument such as in Berge [1, pp. 115–116]. Moreover, by the same assumptions, and noting that there exists some $\delta > 0$ such that $r \geqslant \delta$ on $[\epsilon, s]$, f is bounded. Finally, to show the Lipschitz condition, take arbitrary $x \in [\epsilon, s]$, $v_1, v_2 \in F$, and assume, without loss of generality, that $f(x, v_1) \geqslant f(x, v_2)$. Suppose a_1 maximizes the argument in $f(x, v_1)$. Then

$$\begin{aligned}
&|f(x, v_1) - f(x, v_2)| \\
&\quad\leqslant f(x, v_1) - r^{-1}(x, a_1)\left\{\int [v_2(y) - v_2(x)]\,\beta(x, a_1, dy) - \lambda v_2(x) + p(x, a_1)\right\} \\
&\quad= r^{-1}(x, a_1)\left\{\int [v_1(y) - v_2(y)]\,\beta(x, a_1, dy)\right. \\
&\qquad\left. - [v_1(x) - v_2(x)] \int \beta(x, a_1, dy) - \lambda[v_1(x) - v_2(x)]\right\} \\
&\quad\leqslant r^{-1}(x, a_1)\left\{\sup_{x \leqslant y \leqslant s} |v_1(y) - v_2(y)| \int \beta(x, a_1, dy)\right. \\
&\qquad\left. + \left[\lambda + \int \beta(x, a_1, dy)\right] |v_1(x) - v_2(x)|\right\} \\
&\quad\leqslant L \sup_{x \leqslant y \leqslant s} |v_1(y) - v_2(y)|,
\end{aligned}$$

where for L it suffices to take

$$\sup_{\substack{x\in[\epsilon,s]\\ a\in A}} r^{-1}(x, a) \left\{2 \int \beta(x, a, dy) + \lambda\right\},$$

which is finite by the assumptions. ∎

Since $\epsilon > 0$, the solution of (1) on $[\epsilon, s]$ can obviously be extended to $(0, s]$. However, it cannot necessarily be extended to include $x = 0$, because the possibility that $r(0, a) = 0$ for some $a \in A$ has been allowed, in which case the Lipschitz constant L would be unbounded. This presents a difficulty, namely, the solution v might be unbounded in a neighborhood of the origin, or, even if not, the limit of $v(x)$ as $x \downarrow 0$ might not exist. The following lemma will be used to help resolve this difficulty. For the remainder of this paper, define

$$\bar{p} = \sup_{\substack{x\in S\\ a\in A}} |p(x, a)|.$$

LEMMA 6. *Let v satisfy* (1) *on* $(0, s]$ *and* $v(s) = z$ *for some* z. *Suppose* $\bar{x} \in (0, s]$ *is such that* $v(\bar{x}) > \bar{p}/\lambda$ *and* $v(\bar{x}) \geqslant v(x)$ *for all* $x \in [\bar{x}, s]$ (*alternatively,* $v(\bar{x}) < -\bar{p}/\lambda$ *and* $v(\bar{x}) \leqslant v(x)$ *for all* $x \in [\bar{x}, s]$). *Then v is decreasing* (*respectively, increasing*) *on* $(0, \bar{x}]$.

Proof. The first case will be shown, leaving the other to the reader. According to (1), it must be that $v'(\bar{x}) < 0$, in which case $v'(x) < 0$ for all $x \leqslant \bar{x}$ in some neighborhood of $\bar{x}$. Define $x_0 = \inf\{x \in [0, \bar{x}]: v'(t) \leqslant 0 \text{ for all } t \in (x, \bar{x}]\}$. It follows that $x_0 = 0$, because repeating this argument yields $v'(x_0) < 0$. ∎

At this point, if it is known that $r(0, a) > 0$ for all $a \in A$, then it is easy to assert that there exists a solution of (1) on $[0, s]$ satisfying (2). To see this, note that one can take $\epsilon = 0$ in Lemma 5. Then, applying Lemma 6 with $z > \bar{p}/\lambda$ and $\bar{x} = s$ implies the left-hand side of (2) is negative. Similarly, taking $z < -\bar{p}/\lambda$ makes the left-hand side of (2) positive. By Corollary 4, the left-hand side of (2) is continuous with respect to z, so there must exist some $z \in [-\bar{p}/\lambda, \bar{p}/\lambda]$ such that (2) is satisfied.

On the other hand, if it is known that $r(0, a) = 0$ for some $a \in A$, then a much more delicate argument is required. This matter will be the subject of most of the balance of this section.

For each positive number n, let v_n denote the solution of (1) and $v_n(s) = z_n$, where z_n is chosen so that $z_n \in [-\bar{p}/\lambda, \bar{p}/\lambda]$ and

$$\sup_{a\in A} \left\{\int [v_n(y) - v_n(1/n)]\, \beta(1/n, a, dy) - \lambda v_n(1/n) + p(1/n, a)\right\} = 0. \qquad (4)$$

Such a value of z_n exists, by the same kind of argument as was used in the preceding paragraphs to show the existence of a solution of (1) and (2) in the case where $r(0, a) > 0$ for all $a \in A$. The following indicates some properties of these solutions.

LEMMA 7. *For the functions v_n defined above, $-\bar{p}/\lambda \leqslant v_n(x) \leqslant \bar{p}/\lambda$ for $1/n \leqslant x \leqslant s$, and $v_n'(1/n) = 0$.*

Proof. If the first part were not true, then one could take $\bar{x} = \max\{x \in [1/n, s]: |v_n(x)| = \bar{p}/\lambda\}$, apply Lemma 6, and conclude the left-hand side of (4) is nonzero, a contradiction. For the second part, it suffices to observe that, since r is positive at $1/n$, (4) is true if and only if the right-hand side of (1) equals zero with $v = v_n$ and $x = 1/n$. ∎

Without loss of generality, it can be assumed that the sequence $\{z_n\}$ is convergent. Let z denote the corresponding limit, and, for the remainder of this section, let v denote the solution on $(0, s]$ of (1) and $v(s) = z$. The following lemma will show that v can be extended to include $x = 0$; later, it will be shown that v satisfies (2).

LEMMA 8. *The function v defined above satisfies $v(x) \in [-\bar{p}/\lambda, \bar{p}/\lambda]$ for all $x \in (0, s]$. Moreover, $\lim_{x \searrow 0} v(x)$ exists.*

Proof. By Corollary 4 it is apparent that v_n converges pointwise to v on $(0, s]$. Hence the first conclusion of this lemma follows easily from Lemma 7.

For the second part, it suffices to rule out the possibility that $\lim \sup_{x \searrow 0} v(x) > \lim \inf_{x \searrow 0} v(x)$ by showing that if $\{x_i\}$ is any sequence with $x_i \searrow 0$ and $v'(x_i) = 0$ for $i = 1, 2,...,$ then $\{v(x_i)\}$ is convergent. Now, for $x > 0$, $v'(x) = 0$ if and only if

$$\sup_{a \in A} \left\{\int [v(y) - v(x)]\, \beta(x, a, dy) - \lambda v(x) + p(x, a)\right\} = 0;$$

if and only if

$$v(x) \left[-\lambda - \int \beta(x, a, dy)\right] + \int v(y)\, \beta(x, a, dy) + p(x, a) \leqslant 0$$

for all $a \in A$ with equality for some $a \in A$; if and only if

$$v(x) \geqslant \left[\int v(y)\, \beta(x, a, dy) + p(x, a)\right] \Big/ \left[\lambda + \int \beta(x, a, dy)\right]$$

for all $a \in A$ with equality for some $a \in A$; if and only if

$$v(x) = \sup_{a \in A} \left\{\left[\int v(y)\, \beta(x, a, dy) + p(x, a)\right] \Big/ \left[\lambda + \int \beta(x, a, dy)\right]\right\}. \tag{5}$$

Although v is not known to be defined at $x = 0$, due to the hypotheses about β (see also the proof of the following lemma), the term $\int v(y)\, \beta(x, a, dy)$ and thus the argument in the right-hand side of (5) are continuous on $S \times A$. Hence, setting $x = x_i$, it can be concluded that $\{v(x_i)\}$ converges to

$$\sup_{a \in A} \left\{\left[\int v(y)\, \beta(0, a, dy) + p(0, a)\right] \Big/ \left[\lambda + \int \beta(0, a, dy)\right]\right\}, \tag{6}$$

thereby completing this proof. ∎

The observation made in the following lemma is the key to showing v satisfies (2).

LEMMA 9. *For the sequence* $\{v_n\}$ *defined above, the sequence* $\{v_n(1/n)\}$ *converges to expression* (6).

Proof. By Lemma 7, $v_n'(1/n) = 0$. Hence an equation similar to (5) holds, namely,

$$v_n(1/n) = \sup_{a \in A} \left\{ \left[\int v_n(y)\, \beta(1/n, a, dy) + p(1/n, a) \right] \Big/ \left[\lambda + \int \beta(1/n, a, dy) \right] \right\}.$$

Comparing this with (6), it is apparent that to complete this proof it suffices to show

$$\int v_n(y)\, \beta(1/n, a, dy) \to \int v(y)\, \beta(0, a, dy)$$

as $n \to \infty$. This is accomplished by examining the following inequality (with $0 < \epsilon < s$):

$$\begin{aligned} &\left| \int v_n(y)\, \beta(1/n, a, dy) - \int v(y)\, \beta(0, a, dy) \right| \\ &\quad \leqslant \left| \int_{[0,\epsilon)} (v_n(y) - v(y))\, \beta(1/n, a, dy) \right| + \left| \int_{[\epsilon, s]} (v_n(y) - v(y))\, \beta(1/n, a, dy) \right| \\ &\qquad + \left| \int v(y)\, \beta(1/n, a, dy) - \int v(y)\, \beta(0, a, dy) \right|. \end{aligned} \tag{7}$$

Now $b(1/n, a, [0, 1/n)) = 0$, so by Lemma 7 the first term on the right-hand side is bounded by $2(\bar{p}/\lambda)\, \beta(1/n, a, [0, \epsilon))$. Since $\beta(0, a, \{0\}) = 0$ by hypothesis, $\beta(1/n, a, [0, \epsilon)) \to \beta(0, a, [0, \epsilon))$ as $n \to \infty$ by weak continuity. Hence, the first term can be made arbitrarily small by choosing ϵ small enough and n large enough. With this choice of ϵ, the second term then converges to zero as $n \to \infty$ by Corollary 4. Of course, the third term converges to zero by the weak continuity of β. It follows that the left-hand side of (7) converges to zero as $n \to \infty$, thereby implying the desired result. ∎

LEMMA 10. *The function* v *defined above satisfies* (1) *and* (2) *on* $[0, s]$.

Proof. By Lemma 8, it suffices to verify (2). By the proof of Lemma 8, (2) holds if and only if $v(0)$ equals expression (6). Hence, to complete this proof, it will be shown that if $v(0)$ does not equal expression (6), then a contradiction is obtained.

Suppose $v(0)$ is strictly greater than expression (6) (the opposite case, being similar, will be omitted). Then by (1) and the same kind of equivalences as in the proof of Lemma 8, there exists some $\delta > 0$ such that $v'(x) < -\delta$ for all x in some neighborhood of 0. Now consider the graph of v and the geometry of this situation. By Lemma 9, $v_n(1/n)$ converges to expression (6) as $n \to \infty$. In addition, by Corollary 4, for any $\epsilon > 0$, $v_n \to v$ uniformly on $[\epsilon, s]$ as $n \to \infty$. By choosing ϵ small enough, it follows that there must exist some sequence $\{x_n\}$ converging to zero with $x_n > 1/n$, $v_n'(x_n) = 0$ for all large enough n, and $\{v_n(x_n)\}$ converging to $v(0)$. However, repeating the proof of Lemma 9 with x_n in place of $1/n$, one concludes $\{v_n(x_n)\}$ converges to expression (6), which is the desired contradiction. ▮

Proof of Theorem 1. In view of the preceding lemmas, it remains to show uniqueness. To do this, use will be made of the proposition in the last section, namely, a function v satisfies (1) and (2) if and only if it satisfies (2) and

$$\lambda v(x) = \sup_{a \in A} \left\{ \int [v(y) - v(x)]\, \beta(x, a, dy) - r(x, a)\, v'(x) + p(x, a) \right\}, \qquad x > 0. \tag{8}$$

Now v is one solution of (1) and (2); suppose the continuous function u is another. Let $\bar{x} \in [0, s]$ be such that $| v(\bar{x}) - u(\bar{x})| = \sup_{0 \leqslant x \leqslant s} | v(x) - u(x)|$. Without loss of generality, assume $v(\bar{x}) \geqslant u(\bar{x})$. Then there are three cases:

Case 1. $\bar{x} = s$: This implies $v'(s) \geqslant u'(s)$. Using (8) yields

$$\begin{aligned} 0 &\leqslant \lambda v(s) - \lambda u(s) \\ &= \sup_{a \in A}\{-r(s, a)\, v'(s) + p(s, a)\} - \sup_{a \in A}\{-r(s, a)\, u'(s) + p(s, a)\} \\ &\leqslant -r(s, \bar{a})\, v'(s) + p(s, \bar{a}) + r(s, \bar{a})\, u'(s) - p(s, \bar{a}) \\ &= -r(s, \bar{a})[v'(s) - u'(s)] \leqslant 0, \end{aligned}$$

where $\bar{a}$ maximizes $-r(s, a)\, v'(s) + p(s, a)$ over A. Hence it must be that $v(s) = u(s)$.

Case 2. $0 < \bar{x} < s$: This implies $v'(\bar{x}) = u'(\bar{x})$ and $v(\bar{x}) - u(\bar{x}) \geqslant v(x) - u(x)$ for all $x \in S$. Using (8) yields

$$\begin{aligned} 0 &\leqslant \lambda v(\bar{x}) - \lambda u(\bar{x}) \\ &= \sup_{a \in A} \left\{ \int [v(y) - v(\bar{x})]\, \beta(\bar{x}, a, dy) - r(\bar{x}, a)\, v'(\bar{x}) + p(\bar{x}, a) \right\} \\ &\qquad - \sup_{a \in A} \left\{ \int [u(y) - u(\bar{x})]\, \beta(\bar{x}, a, dy) - r(\bar{x}, a)\, u'(\bar{x}) + p(\bar{x}, a) \right\} \\ &\leqslant \int [v(y) - v(\bar{x})]\, \beta(\bar{x}, \bar{a}, dy) - r(\bar{x}, \bar{a})\, v'(\bar{x}) + p(\bar{x}, \bar{a}) \\ &\qquad - \int [u(y) - u(\bar{x})]\, \beta(\bar{x}, \bar{a}, dy) + r(\bar{x}, \bar{a})\, u'(\bar{x}) - p(\bar{x}, \bar{a}) \\ &= \int [(v(y) - u(y)) - (v(\bar{x}) - u(\bar{x}))]\, \beta(\bar{x}, \bar{a}, dy) \leqslant 0, \end{aligned}$$

where $\bar{a}$ maximizes the argument in (8) with $x = \bar{x}$. Again, it must be that $v(\bar{x}) = u(\bar{x})$.

Case 3. $\bar{x} = 0$: This implies $v(0) - u(0) \geqslant v(x) - u(x)$ for all $x \in S$. Using (2) yields

$$\begin{aligned} 0 &\leqslant \lambda v(0) - \lambda u(0) \\ &= \sup_{a \in A} \left\{ \int [v(y) - v(0)]\, \beta(0, a, dy) + p(0, a) \right\} \\ &\quad - \sup_{a \in A} \left\{ \int [u(y) - u(0)]\, \beta(0, a, dy) + p(0, a) \right\} \\ &\leqslant \int [(v(y) - u(y)) - (v(0) - u(0))]\, \beta(0, \bar{a}, dy) \leqslant 0, \end{aligned}$$

and this case is disposed of in the same way as the first two. ∎

4. Proof of Theorem 2

Attention is now given to the problem of showing there exists a solution to (1) and (2) in the case where $S = [0, \infty)$. The basic approach will be to first construct a sequence of functions that satisfy (1) and (2) on a corresponding sequence of compact intervals and then to show this sequence converges to a function which satisfies (1) and (2) on all of $[0, \infty)$.

With β defined on $S = [0, \infty)$ satisfying the four assumptions stated in the Introduction, for each real number $s > 0$ define a new measure β_s on $[0, s]$ in the following manner:

$$\beta_s(x, a, dy) = \begin{cases} \beta(x, a, dy), & x < s, \quad y < s, \\ \beta(x, a, [s, \infty)), & x \leqslant s, \quad y = s, \\ 0, & \text{otherwise}. \end{cases}$$

This new measure clearly satisfies assumptions (i)–(iii) stated in the Introduction for the case $S = [0, s]$. Moreover, β_s satisfies the weak continuity assumption. To see this, let u be an arbitrary bounded, continuous function on $[0, s]$. Define the continuous function $\bar{u}$ on $[0, \infty)$ by

$$\bar{u}(x) = \begin{cases} u(x), & x < s, \\ u(s), & x \geqslant s. \end{cases}$$

Then

$$\begin{aligned} \int u(y)\, \beta_s(x, a, dy) &= \int_{[0,s)} \bar{u}(y)\, \beta(x, a, dy) + \int_{\{s\}} \bar{u}(y)\, \beta_s(x, a, dy) \\ &= \int \bar{u}(y)\, \beta(x, a, dy), \end{aligned}$$

which is continuous with respect to (x, a) by the weak continuity of β.

Hence the hypotheses of Theorem 1 are satisfied for each fixed $s > 0$ and the corresponding measure β_s, so let v_s denote the corresponding solution of (1) and (2) on $[0, s]$. The objective here will be to show that, as $s \to \infty$, the sequence $\{v_s\}$ converges in a natural way to some function which is the unique solution of (1) and (2) on $S = [0, \infty)$.

The first step will be to show $\{v_s(s)\}$ converges to $\bar{q}/\lambda$ as $s \to \infty$, where

$$\bar{q} = \sup_{a \in A} |q(a)|$$

and q is defined in Theorem 2. Let $\epsilon > 0$ be arbitrary. Let x_ϵ be such that $|p(x, a) - q(a)| < \epsilon$ for all $x \geqslant x_\epsilon$ and $a \in A$. Define

$$K \equiv \inf_{\substack{x \in [x_\epsilon, \infty) \\ a \in A}} r^{-1}(x, a);$$

recall $K > 0$ by the hypothesis of Theorem 2.

Consider the ordinary, linear, differential equation

$$y'(x) = K[-\lambda y(x) + \bar{q} + \epsilon], \qquad y(x_\epsilon) = \bar{p}/\lambda;$$

where, as in the preceding section, $\bar{p} \equiv \sup\{|p(x, a)|: x \in [0, \infty), a \in A\}$. The solution is of the form

$$y(x) = Ce^{-K\lambda x} + (\bar{q} + \epsilon)/\lambda$$

for some constant C. Without loss of generality $\bar{p} > \bar{q} + \epsilon$, so it must be that $C > 0$. This solution is used in the following.

LEMMA 11. *For arbitrary $\epsilon > 0$, let x_ϵ and y be defined as above. Let $s > x_\epsilon$. Then $v_s(x) \leqslant y(x)$ for all $x \in [x_\epsilon, s]$.*

Proof. In view of Lemma 8, it must be that $v_s(x_\epsilon) \leqslant y(x_\epsilon)$, so it suffices to show that $v_s(\bar{x}) \geqslant y(\bar{x})$ for some $\bar{x} \in (x_\epsilon, s]$ implies $v_s(x) > y(x)$ for all $x \in [x_\epsilon, \bar{x})$.

Now $C \geqslant 0$, so y is decreasing and, without loss of generality, one can assume $v_s(\bar{x}) \geqslant v_s(x)$ for all $x \in [\bar{x}, s]$. By (1),

$$\begin{aligned} v_s'(\bar{x}) &= \sup_{a \in A} \left\{ r^{-1}(\bar{x}, a) \left[\int [v_s(y) - v_s(\bar{x})]\, \beta(\bar{x}, a, dy) - \lambda v_s(\bar{x}) + p(\bar{x}, a) \right] \right\} \\ &< r^{-1}(\bar{x}, \bar{a})[-\lambda v_s(\bar{x}) + \bar{q} + \epsilon] \\ &\leqslant K[-\lambda y(\bar{x}) + \bar{q} + \epsilon] = y'(\bar{x}), \end{aligned}$$

where $\bar{a}$ is chosen to maximize the above argument, the first inequality follows from $v_s(y) - v_s(\bar{x}) \leqslant 0$ and $p(\bar{x}, \bar{a}) < \bar{q} + \epsilon$, and the second inequality follows from $v_s(\bar{x}) \geqslant y(\bar{x})$ and the definition of K. By continuity, it follows that $v_s'(x) < y'(x)$ for all $x < \bar{x}$ in some neighborhood of $\bar{x}$. Define

$$x_0 \equiv \inf\{x \in [x_\epsilon, \bar{x}]: v_s'(t) \leqslant y'(t) \text{ for all } t \in [x, \bar{x}]\}.$$

It must be that $x_0 = x_\epsilon$, or else one could repeat this argument with $\bar{x} = x_0$.

LEMMA 12. *For any* $\delta > 0$, *there exists some* $x_\delta > 0$ *such that, for any* $s > x_\delta$, $|v_s(x) - \bar{q}/\lambda| < \delta$ *for all* $x \in [x_\delta, s]$.

Proof. In exactly the same way as Lemma 11, for any $\epsilon > 0$ and the same x_ϵ, there exist some negative constant D and some function $z(x) \equiv De^{-K\lambda x} + (\bar{q} - \epsilon)/\lambda$ such that, for any $s > x_\epsilon$, $v_s(x) \geqslant z(x)$ for all $x \in [x_\epsilon, s]$. First set $\epsilon = \lambda\delta/4$, and then set $x_\delta > x_\epsilon$ such that $y(x_\delta) - z(x_\delta) \leqslant \delta$. The desired result follows immediately. ▮

For each $s > 0$, define the function on $[0, \infty)$

$$\bar{v}_s(x) = \begin{cases} v_s(x), & x < s, \\ v_s(s), & x \geqslant s. \end{cases}$$

It will turn out that the sequence $\{\bar{v}_s\}$ converges uniformly to the solution of (1) and (2).

LEMMA 13. *The sequence* $\{\bar{v}_s\}$ *converges uniformly to some function* v *which is bounded and continuous on* $[0, \infty)$.

Proof. It suffices to show that the sequence $\{\bar{v}_s\}$ is Cauchy under the supremum norm $\|\cdot\|$ on $[0, \infty)$. Suppose $s < t$. By Lemma 12, $\sup_{x \geqslant s} |\bar{v}_s(x) - \bar{v}_t(x)|$ can be made arbitrarily small by choosing large enough s, so this Cauchy property will follow immediately from the following two cases:

Case 1. $|\bar{v}_s(0) - \bar{v}_t(0)| = \|\bar{v}_s - \bar{v}_t\|$: Assume $\bar{v}_s(0) \geqslant \bar{v}_t(0)$, leaving the opposite case to the reader. Using (2) and the definition of β_s, one obtains:

$$\begin{aligned}
0 &\leqslant \lambda\bar{v}_s(0) - \lambda\bar{v}_t(0) \\
&= \sup_{a\in A} \left\{\int [v_s(y) - v_s(0)]\, \beta_s(0, a, dy) + p(0, a)\right\} \\
&\quad - \sup_{a\in A} \left\{\int [v_t(y) - v_t(0)]\, \beta_t(0, a, dy) + p(0, a)\right\} \\
&\leqslant \int [v_s(y) - v_s(0)]\, \beta_s(0, \bar{a}, dy) - \int [v_t(y) - v_t(0)]\, \beta_t(0, \bar{a}, dy) \\
&= \int_{[0,s]} [(v_s(y) - v_t(y)) - (v_s(0) - v_t(0))]\, \beta(0, \bar{a}, dy) \\
&\quad + [v_s(s) - v_s(0)]\, \beta(0, \bar{a}, (s, \infty)) - \int_{(s,t]} [v_t(y) - v_t(0)]\, \beta_t(0, \bar{a}, dy) \\
&\leqslant \int_{(s,t]} [(v_s(s) - v_t(y)) - (v_s(0) - v_t(0))]\, \beta_t(0, \bar{a}, dy) \\
&\leqslant \int_{(s,t]} [v_s(s) - v_t(y)]\, \beta_t(0, \bar{a}, dy),
\end{aligned}$$

where $\bar{a}$ maximizes the first argument after the first equality. Here the next to last inequality uses the identity $\beta(0, \bar{a}, (s, \infty)) = \beta_t(0, \bar{a}, (s, t])$ as well as the fact that $v_s(0) - v_t(0) \geqslant v_s(x) - v_t(x)$ for all $x \in [0, s]$. Hence, by Lemma 12 and assumption (iii) about β, $| \bar{v}_s(0) - \bar{v}_t(0)|$ can be made arbitrarily small by choosing large enough s.

Case 2. $| \bar{v}_s(\bar{x}) - \bar{v}_t(\bar{x})| = \| \bar{v}_s - \bar{v}_t \|$ for some $\bar{x} \in (0, s)$. Assume $v_s(\bar{x}) \geqslant v_t(\bar{x})$, leaving the opposite case to the reader. Using (8), and with $\bar{a}$ maximizing the first argument after the first equality, one obtains

$$\begin{aligned}
0 &\leqslant \lambda\bar{v}_s(\bar{x}) - \lambda\bar{v}_t(\bar{x}) \\
&= \sup_{a\in A} \left\{\int [v_s(y) - v_s(\bar{x})]\, \beta(\bar{x}, a, dy) - r(\bar{x}, a)\, v_s'(\bar{x}) + p(\bar{x}, a)\right\} \\
&\quad - \sup_{a\in A} \left\{\int [v_t(y) - v_t(\bar{x})]\, \beta(\bar{x}, a, dy) - r(\bar{x}, a)\, v_t'(\bar{x}) + p(\bar{x}, a)\right\} \\
&\leqslant \int [v_s(y) - v_s(\bar{x})]\, \beta_s(\bar{x}, \bar{a}, dy) - \int [v_t(y) - v_t(\bar{x})]\, \beta_t(\bar{x}, \bar{a}, dy),
\end{aligned}$$

where the last inequality uses the identity $\bar{v}_s'(\bar{x}) - \bar{v}_t'(\bar{x}) = 0$. From this point one proceeds in exactly the same way as in the first case and concludes that $| \bar{v}_s(\bar{x}) - \bar{v}_t(\bar{x})|$ can be made arbitrarily small by choosing large enough s.

LEMMA 14. *Let $v \equiv \lim_{s\to\infty} \bar{v}_s$ be as in Lemma* 13. *Then v satisfies* (1) *and* (2).

Proof. First of all, $\bar{v}_s$ satisfies boundary condition (2), so letting $s \to \infty$ and using Lemma 13 one concludes v satisfies (2). To show that it satisfies (1), let $\epsilon > 0$ be fixed so that, for any $x \geqslant \epsilon$,

$$v_s(x) = v_s(\epsilon) + \int_\epsilon^x v_s'(t)\, dt. \tag{9}$$

Since v_s satisfies (1),

$$v_s'(t) \to \sup_{a\in A} \left\{r^{-1}(t, a) \left[\int [v(y) - v(t)]\, \beta(t, a, dy) - \lambda v(t) + p(t, a)\right]\right\}$$

as $s \to \infty$ for each fixed $t \in [\epsilon, x]$. This limit is bounded on $[\epsilon, x]$, so letting $s \to \infty$ in (9) and using the bounded convergence theorem, one concludes

$$\begin{aligned}
v(x) = v(\epsilon) + \int_\epsilon^x \sup_{a\in A} \Big\{r^{-1}(t, a) \Big[\int [v(y) - v(t)]\, \beta(t, a, dy)& \\
- \lambda v(t) + p(t, a)\Big]\Big\}\, dt.&
\end{aligned}$$

This and the arbitrary choice of ϵ imply v satisfies (1). ∎

Proof of Theorem 2. All of this follows from earlier lemmas with the exception of uniqueness. The proof of the latter will only be sketched, since it is similar to the analogous part of Theorem 1.

Now v is one bounded, continuous solution of (1) and (2); suppose u is another. If there exists some $\bar{x} \geqslant 0$ such that $|v(\bar{x}) - u(\bar{x})| = \sup_{x \geqslant 0} |v(x) - u(x)|$, then one can proceed as in cases 2 and 3 in the uniqueness proof of Theorem 1 to show that, in fact, $v(\bar{x}) = u(\bar{x})$. The only other possibility is that $\sup_{x \geqslant 0} |v(x) - u(x)| = \limsup_{x \to \infty} |v(x) - u(x)|$. If this quantity is nonzero, then by the same kind of thinking as Lemma 11 it would mean that $u(x) \notin [-\bar{p}/\lambda, \bar{p}/\lambda]$ for some x. By Lemma 6, that would imply $u(0) \notin [-\bar{p}/\lambda, \bar{p}/\lambda]$, which, in turn, would imply u does not satisfy (2), a contradiction. ∎

5. The Markov Control Problem

The functional differential equations studied in this paper are motivated by the problem of controlling a storage process. Such processes have been studied extensively by Moran [11], Çinlar and Pinsky [4], Çinlar [3], Harrison and Resnick [8], and others. Briefly, a storage process is a Markov process with a state space such as the interval S defined above, an input process having nondecreasing right continuous paths of the pure jump type, and a state dependent release rate. Since storage processes have applications in the theories of dams, queues, and the economics of natural resources, it is not inappropriate to consider controlled generalizations of these processes.

Consider the problem of controlling a stationary version of a storage process in such a manner as to maximize the expected discounted reward over an infinite time horizon. Abstract versions of this problem for general Markov processes have been studied by Vermes [12] and Doshi [5]. Their approach is to formulate a dynamic programming functional equation in terms of the weak infinitesimal generator of the process. They show that if a solution exists, then a variety of conclusions readily follow, including that the solution equals the maximum expected discounted reward. Unfortunately, there are no results on the existence of a solution in the general case; instead, different approaches are apparently necessary for each specific kind of Markov process. In particular, the desire to show that a solution exists to the storage process functional equation led to Theorems 1 and 2 above. The balance of this section will briefly explain why this functional equation is equivalent to (1) and (2).

The state space for the process is the interval S; either the compact case $S = [0, s]$ or the unbounded case $S = [0, \infty)$ will do. The set A is the set of admissible actions. Let D denote the set of all bounded and continuous functions f on S for which the left-hand derivative $df^{-}(x)/dx$ exists for all $x > 0$. The first step is to define, for each $a \in A$, a map $\mathcal{A}(a)$ on D such that $\mathcal{A}(a)$ is the weak infinitesimal generator of a storage process. Morais [10] shows that for $\mathcal{O}\!\mathcal{l}(a)$ one should take

$$\mathcal{O}\!\mathcal{l}(a) f(x) = -\frac{df^{-}(x)}{dx} r(x, a) + \int [f(y) - f(x)]\, \beta(x, a, dy), \qquad x > 0,$$
$$\mathcal{O}\!\mathcal{l}(a) f(0) = \int [f(y) - f(0)]\, \beta(0, a, dy).$$

Here r and β are defined as earlier, only now they have the following interpretations: The value $r(x, a)$ specifies the release rate of the process at any instant when the state is x and action a is used. The measure $\beta(x, a, dy)$ describes the input process when the storage process is in state x and action a is used. Specifically, $\beta(x, a, S)$ equals the input arrival rate and $\beta(x, a, dy)/\beta(x, a, S)$ equals the conditional distribution of the state of the process given an input just arrived, the state of the process just before the arrival was x, and action a was used. A detailed justification for this interpretation may be found in Morais [10].

Having defined the generator $\mathcal{O}\!\mathcal{l}$, one may proceed as in Doshi [5] or Vermes [12] and write down the dynamic programming functional equation for this problem, namely,

$$\sup_{a \in A}\{\mathcal{O}\!\mathcal{l}(a) f(x) + p(x, a)\} - \lambda f(x) = 0, \qquad x \geqslant 0,$$

where λ is the discount rate and p can be interpreted as the payoff rate as a function of the state and action. Note that this equation is equivalent to (8) and (2). Hence to show there exists some $v \in D$ which satisfies this equation, it is sufficient by Theorems 1 and 2 to show that (1) and (8) are equivalent. This is accomplished by the following.

PROPOSITION 15. *A bounded, differentiable function v satisfies* (1) *if and only if it satisfies* (8).

Proof. For any fixed x, the arguments in both (1) and (8) are continuous with respect to a, so the suprema are always attained. It suffices to observe the following statements are equivalent for each fixed x:

(i) (1) is satisfied.

(ii) $v'(x) \geqslant r^{-1}(x, a)[\int [v(y) - v(x)]\, \beta(x, a, dy) - \lambda v(x) + p(x, a)]$ for all $a \in A$ with equality for some $a \in A$.

(iii) $0 \geqslant r^{-1}(x, a)[\int [v(y) - v(x)]\, \beta(x, a, dy) - r(x, a)\, v'(x) - \lambda v(x) + p(x, a)]$ for all $a \in A$ with equality for some $a \in A$.

(iv) $0 \geqslant \int [v(y) - v(x)]\, \beta(x, a, dy) - r(x, a)\, v'(x) - \lambda v(x) + p(x, a)$ for all $a \in A$ with equality for some $a \in A$.

(v) (8) is satisfied. ∎

Note that to formulate and analyse the functional equation there was no need to specify the admissible controls. Morais [10] does this as well as address other important aspects of the storage process control problem. In fact, his work together with this one comprise a fairly complete picture of the theory of storage process control in the case of discounted rewards over an infinite time horizon.

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